A COMPLETE CHARACTERIZATION OF R-SETS IN THE THEORY OF DIFFERENTIATION OF INTEGRALS

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ABSTRACT. Let $\mathcal{R}_s$ be the family of open rectangles in the plane R^2 having slope s with the abscissa. We say a set of slopes S is R-set if there exists a function $f \in L(R^2)$, such that the basis $\mathcal{R}_s$ differentiates integral of f if $s \notin S$, and

$$\overline{D}_s f(x) = \limsup_{\operatorname{diam}(R)\to 0, x\in R\in\mathcal{R}_s} \frac{1}{|R|}\int_R f = \infty$$

almost everywhere if $s \in S$. If the condition $\overline{D}_s f(x) = \infty$ holds on a set of positive measure (instead of a.e.) we shall say it is WR-set. It is proved, that S is a R-set(WR-set) if and only if it is $G_\delta(G_{\delta\sigma})$.

1. Introduction

For any number $s \in [0, \frac{\pi}{2})$ we define $\mathcal{R}_s$ to be the family of all open rectangles R in $\mathbb{R}^2$ having slope s, i.e. R has a side forming angle s with the abscissa. We say that the basis $\mathcal{R}_s$ differentiates the integral of the function $f \in L^1(\mathbb{R}^2)$, if

$$\lim_{d(R)\to 0, x\in R\in\mathcal{R}_s} \frac{1}{|R|}\int_R f = f(x) \tag{1.1}$$

almost everywhere in $\mathbb{R}^2$, where $d(R)$ is the diameter of R. According to the well-known theorem of Jessen-Marcinkiewicz-Zygmund [3] the basis $\mathcal{R}_s$ differentiates $\int f$ for any function $f \in L\log L(\mathbb{R}^2)$. On the other hand S. Saks [12] constructed an example of function $f \in L^1(\mathbb{R}^2)$ such that

$$\overline{D}_s f(x) = \limsup_{d(R)\to 0, x\in R\in\mathcal{R}_s} \frac{1}{|R|}\int_R f = \infty, \text{ everywhere .}$$

In view of this A. Zygmund in [1] posed the following problem: for a given $f \in L^1(\mathbb{R}^2)$ is it possible to find a direction s such that $\mathcal{R}_s$ differentiates $\int f$? J. Marstrand in [7] gave a negative answer to this question, proving

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Theorem (J. Marstrand)**.** *There exists a function $f \in L^1(\mathbb{R}^2)$ such that $\overline{D}_s f(x) = \infty$ almost everywhere for any s.*

Different generalizations of this result are obtained by J. El Helou [2], A. M. Stokolos [13], B. López Melero [6] and G. G. Oniani [9]. A. M. Stokolos in [13] extended Marstrand's theorem to higher dimensional case. In the papers [6] and [9] it is considered the same problem for general translation invariant differentiation basises.

We say that the set $S \subset [0, \frac{\pi}{2})$ is R-set if there exists a function $f \in L^1(\mathbb{R}^2)$ such that the basis $\mathcal{R}_s$ differentiates $\int f$ whenever $s \in [0, \pi/2) \setminus S$, and $\overline{D}_s f(x) = \infty$ almost everywhere as $s \in S$. If the condition $\overline{D}_s f(x) = \infty$ holds on a set of positive measure (instead of a.e.) we shall say it is WR-set (weak R-set). In this language, Marstrand's theorem asserts, that $[0, \pi/2)$ is R-set. A. M. Stokolos in [14] proved, the existence of everywhere dense WR-set, which is not whole $[0, \pi/2)$. G. Lepsveridze in [4],[5] proved that any finite set is R-set and any countable set is in some WR-set of measure zero. G. G. Oniani in [9] generalizing this result proved that any countable set is in some R-set of measure zero.

The definition of R-sets first appeared in the paper [8] by G. G. Oniani, where the author posed the problem about characterization of all R-sets. In particular, it was a question if there exists a R-set of positive measure and moreover whether any interval is R-set or not? In the same paper Oniani shows, that any R-set is G_δ in $[0, \pi/2)$, i.e.

$$G = \left(\cap_{k=1}^{\infty} G_k \right) \cap [0, \pi/2)$$

where G_n are open sets, and conversely if G_δ-set is countable, then it is R-set. These results characterize the countable R-sets. We note that any countable G_δ-set is nowhere dense. So in [8] Oniani constructed also a R-set of second category. These problems are stated also in the monograph G. G. Oniani [9] and in the papers [10] and [11] it is investigated the higher dimensional case of the problem.

The following theorems give a complete characterization of general R and WR-sets.

Theorem 1. *For the set $S \subset [0, \pi/2)$ to be R-set it is necessary and sufficient to be G_δ.*

Theorem 2. *For the set $S \subset [0, \pi/2)$ to be WR-set it is necessary and sufficient to be $G_{\delta\sigma}$.*

The necessity of Theorem 1 is proved by Oniani in [8]. We present here a short statement of the proof of that. If S is a R-set, then there exists a function $f \in L^1$ such that (1.1) holds as $s \in [0, \pi/2) \setminus S$ and $\overline{D}_s f(x) = \infty$ a.e. as $s \in S$. For any $n \in \mathbb{N}$ denote

$$U_n = \{s \in [0, \pi/2) : |\{x \in B(n),\, M_s^{[0,1/n)} f(x) > n\}| > |B(n)| - 2^{-n}\},$$

where $B(n) = \{x \in \mathbb{R}^2 : \|x\| \leq n\}$ and the maximal function $M_s f$ are defined in Section 2. It is easy to check, that $U_n = G_n \cap [0, \pi/2)$, where G_n are open sets and

$$\{s \in [0, \pi/2) : \overline{D}_s f(x) = \infty \text{ a.e. }\} = \bigcap_n U_n = \left(\bigcap_n G_n\right) \bigcap [0, \pi/2),$$

i.e. it is G_δ-set in $[0, \pi/2)$, which proves the one part of Theorem 1.

To prove the necessity of Theorem 2 it is enough to prove that for any function $f \in L^1(\mathbb{R}^2)$ the set

$$G_f = \{s \in [0, \pi/2) : |\{x \in \mathbb{R}^2 : \overline{D}_s f(x) = \infty\}| > 0\}$$

is $G_{\delta\sigma}$. Denote

$$U_{nm} = \{s \in [0, \pi/2) : |\{x \in B(n) : M_s^{[0,1/m)} f(x) > m\}| > \frac{1}{n}\}, \quad n, m = 1, 2, \cdots,$$

where $B(n)$ and $M_s f$ are defined in Section 2. It is clear U_{nm} are open sets in $[0, \pi/2)$ and

$$G_f = \bigcup_n \bigcap_m U_{nm}.$$

To show the last equality it suffices to check the following relations:

$$\begin{aligned} s \in G_f &\Leftrightarrow |\{x \in \mathbb{R}^2 : \overline{D}_s f(x) = \infty\}| > \alpha > 0 \\ &\Leftrightarrow \exists\, n \text{ such that } |\{x \in B_n : \overline{D}_s f(x) = \infty\}| > \frac{1}{n} \\ &\Leftrightarrow \exists\, n \text{ such that } s \in \bigcap_m U_{n,m} \Leftrightarrow s \in \bigcup_n \bigcap_m U_{n,m}. \end{aligned}$$

Hence the set G_f is $G_{\delta\sigma}$.

We shall prove the sufficiencies of the theorems invoking the probabilistically independence of sets similar to original approach of J. Marstrand in [7]. This idea is involved in Lemma 1. Of coarse, we use also Bohr's construction displayed in Saks's classical counterexample. It is important that the function constructed in the proof is not nonnegative, which we don't

have in all the results stated above. This argument gives more freedom in the construction to ensure differentiability of the integral along some directions. So the method demonstrated in the proof differs from the others, because we essentially use an interference of positive and negative values of a function in integrals, which is displayed in Lemma 2 and Lemma 3.

2. Notations and Lemmas

The basis $\mathcal{R}_s$ can be defined for any $s \in [0, 2\pi]$. We note that $\mathcal{R}_s = \mathcal{R}_t$ if $s = t \mod \pi/2$. In fact $\cup_{s\in[0,\pi/2)}\mathcal{R}_s$ is the family of all rectangles in the plane.
If $n \in \mathbb{N}$ is an integer and $c = (c_1, c_2)$, then for any set $A \subset \mathbb{R}^2$ we denote

$$\operatorname{dil}_n A = \{x = (x_1, x_2) \in \mathbb{R}^2 : nx = (nx_1, nx_2) \in A\},$$
$$c + A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x = c + a,\, a \in A\}.$$

We let $Q_0 = [-1/2, 1/2) \times [-1/2, 1/2)$ and for any $n \in \mathbb{N}$, $k = (k_1, k_2) \in \mathbb{Z}^2$ denote $Q_k^n = \operatorname{dil}_n\big(k + Q_0\big)$. For a fixed n the family $\{Q_k^n : k \in \mathbb{Z}^2\}$ is a partition of the plane to squares with side lengths $1/n$. In some places for Q_k^1 we shall use simply Q_k.
We denote by $\operatorname{rot}_s A$ the rotation of the set $A \subset \mathbb{R}^2$ round the point $(0, 0)$ by angle s. Denote $B(\varepsilon) = \{x \in \mathbb{R}^2 : \|x\| = \sqrt{x_1^2 + x_2^2} \le \varepsilon\}$ and $\Gamma_s(\varepsilon) = \operatorname{rot}_s\{x = (x_1, x_2) : |x_2| < \varepsilon\}$.
The notation $s^\perp$ stands for the direction $s + \pi/2$. For any direction s define $\operatorname{mes}_s A$ to be the linear Lebesgue measure of the projection of A on the line parallel to $s^\perp$.
For any measurable set $A \subset \mathbb{R}^2$ we denote

$$\operatorname{mes}^* A = \sup_{k\in\mathbb{Z}^2} |A \cap Q_k|,$$
$$\operatorname{mes}_* A = \inf_{k\in\mathbb{Z}^2} |A \cap Q_k|.$$

For numbers $0 < \delta < \mu \le \infty$ we define $\mathcal{R}_s^{[\delta,\mu)}$ to be the family of rectangles $R = R_1 \times R_2 \in \mathcal{R}_s$ with $\delta \le |R_1|, |R_2| < \mu$ and we let $\mathcal{R}_s^\delta$ to be the rectangles from $\mathcal{R}_s$ with $|R_1| = |R_2| = \delta$. Denote

$$M_s^{[\delta,\mu)} f(x) = \sup_{R\in\mathcal{R}_s^{[\delta,\mu)}} \frac{1}{|R|}\left|\int_R f(x)dx\right|.$$

If $\delta = 0$ and $\mu = \infty$ we shall use notation $M_s f(x)$. We say that the set $A \subset \mathbb{R}^2$ is δ-set if it is a union of mutually disjoint rectangles from the family $\mathcal{R}_s^{[\delta,\infty)}$. The following lemma contains the main idea of the proof of Marstrand's theorem.

Lemma 1. *Suppose* $0<\delta_t<1$, $t=1,2,\cdots,T$ *are arbitrary numbers and* $A_t\subset\mathbb{R}^2$ *are* δ_t*-sets with* $\operatorname{mes}_*(A_t)>12\delta_t$, $t=1,2,\cdots,T$. *Then for any sequence of integers* $\{n_t\}$, $n_1=1$, $n_{t+1}>\frac{4}{\delta_t}n_t$, *we have*

$$\operatorname{mes}_*\left(\bigcup_{t=1}^{T}\operatorname{dil}_{n_t}(A_t)\right)>1-\left(1-\frac{\operatorname{mes}_*(A_t)}{32}\right)^T. \tag{2.1}$$

Proof. First we prove that if B is δ-set with $\operatorname{mes}_*B>12\delta$, $m,n\in\mathbb{N}$ and $n>\frac{4}{\delta}m$, then there exists a set $\tilde{B}$ such that

1) $\tilde{B}\subset\operatorname{dil}_m B$,
2) for any $k\in\mathbb{Z}^2$ the set $\tilde{B}\cap Q_k^m$ is a union of squares Q_j^n,
3) the values $|\tilde{B}\cap Q_k^m|$ are equal for different $k\in\mathbb{Z}^2$,
4) $\operatorname{mes}_*(\tilde{B})>\frac{1}{32}\operatorname{mes}_*B$.

We note that any rectangle $R\in\mathcal{R}_s^{[\delta,\infty)}$ is a union of rectangles from $\mathcal{R}_s^{\delta}$. So we have $\operatorname{dil}_m B=\cup_i R_i$ where $R_i\in\mathcal{R}_s^{\delta/m}$. Denote

$$B'=\bigcup_{R_i\subset Q_k^m\text{ for some }k\in\mathbb{Z}^2}R_i\subset\operatorname{dil}_m B.$$

We have $\operatorname{diam}(R_i)=\frac{\delta\sqrt{2}}{m}$. So if $R_i\not\subset Q_k^m$ then $R_i\cap\tilde{Q}_k^m=\varnothing$ as $k\in\mathbb{Z}^2$, where $\tilde{Q}_k^m$ is the square concentric Q_k^m with side lengths $\frac{1}{m}(1-2\delta\sqrt{2})$. Hence we get

$$\begin{aligned}|B'\cap Q_k^m|&>|\operatorname{dil}_m B\cap Q_k^m|-|Q_k^m\setminus\tilde{Q}_k^m|\\&=|\operatorname{dil}_m B\cap Q_k^m|-\frac{1}{m^2}(4\delta\sqrt{2}-8\delta^2)>|\operatorname{dil}_m B\cap Q_k^m|-\frac{6\delta}{m^2}\\&=\frac{1}{m^2}|B\cap Q_k^1|-\frac{6\delta}{m^2}\ge\frac{1}{m^2}(\operatorname{mes}_*B-6\delta)>\frac{\operatorname{mes}_*B}{2m^2}.\end{aligned}\tag{2.2}$$

Using Besicovitch theorem on covering by squares (see [1], p. 10), we may choose a subfamily $\{R_i'\}$ from $\{R_i\}$ such that R_i' are pairwise disjoin and

$$\left|\bigcup_{R_i'\subset Q_k^m}R_i'\right|\ge\frac{1}{4}\left|\bigcup_{R_i\subset Q_k^m}R_i\right|\quad\text{for any }k\in\mathbb{Z}^2.\tag{2.3}$$

Therefore, denoting

$$B''=\bigcup R_i'\subset B'\subset\operatorname{dil}_m B,$$

by (2.2) and (2.3) we have

$$|B''\cap Q_k^m|>\frac{\operatorname{mes}_*B}{8m^2},\quad k\in\mathbb{Z}^2.\tag{2.4}$$

Using a simple geometry, one can easily check that if $R\in\mathcal{R}_s^{\delta/m}$ and $n>\frac{4}{\delta}m$, then

$$\left|\bigcup_{Q_j^n\subset R}Q_j^n\right|>\frac{1}{4}|R|.$$

So, by virtue of (2.4), for $n>\frac{4}{\delta}m$ we have

$$\left|\bigcup_{Q_j^n\subset B''\cap Q_k^m}Q_j^n\right|>\frac{1}{4}|B''\cap Q_k^m|>\frac{\mathrm{mes}_*B}{32m^2}.$$

Taking away some of the squares Q_j^n from the left union we can get a set $\tilde{B}\subset B''$, which is again a union of the squares Q_j^n and in addition all the sets $\tilde{B}\cap Q_k^m$ consist of a same number of squares Q_j^n and $|\tilde{B}\cap Q_k^m|\ge\frac{\mathrm{mes}_*B}{32m^2}$, $k\in\mathbb{Z}^2$. Certainly, $\tilde{B}$ satisfies the conditions (1)-(4)

Taking $n=n_{t+1}$, $m=n_t$, $B=A_{n_t}$, $t=1,2,\cdots,T$ we get sets $\tilde{A}_t$, $t=1,2,\cdots,T$, such that

1) $\tilde{A}_t\subset \mathrm{dil}_{\,n_t}A_t$,
2) $\tilde{A}_t\cap Q_k^{n_t}$ is a union of squares $Q_j^{n_{t+1}}$ for any $k\in\mathbb{Z}^2$,
3) the values $|\tilde{A}_t\cap Q_k^{n_t}|$ are equal for different $k\in\mathbb{Z}^2$,
4) $\mathrm{mes}_*(\tilde{A}_t)>\frac{\mathrm{mes}_*(A_t)}{32}$.

From the conditions $2),3)$ it follows that for the fixed $k\in\mathbb{Z}^2$ the sets $\tilde{A}_t\cap Q_k$, $t=1,2,\cdots,T$ are probabilistically independent. Then by 1) and 4)

$$\begin{aligned}\mathrm{mes}_*\left(\bigcup_{t=1}^T\mathrm{dil}_{\,n_t}A_t\right)\ge\mathrm{mes}_*\left(\bigcup_{t=1}^T\tilde{A}_t\right)=\left|\bigcup_{t=1}^T(\tilde{A}_t\cap Q_k)\right|\\ =1-\left(1-\mathrm{mes}_*(\tilde{A}_t)\right)^T>1-\left(1-\frac{\mathrm{mes}_*(A_t)}{32}\right)^T.\end{aligned}$$

□

For any line $l\subset\mathbb{R}^2$ we denote by $\arg l$ the positive value of the minimal angle between l and x-axes. For two points θ, $\theta'\in\mathbb{R}^2$ we denote by $\theta\theta'$ the line passing through θ and θ', and by $[\theta,\theta']$ the line segment with vertices θ and θ'.

Lemma 2. *Let $0<\varepsilon<1$, $0<\gamma<\frac{\pi}{12}$ be any numbers and*

$$\theta_k=(\varepsilon/2^k,\ \mathrm{sign}\,(k)\cdot\mathrm{tg}\gamma\cdot\varepsilon/2^k),\ k=\pm1,\pm2,\cdots.\tag{2.5}$$

Then for any rectangle $R\in\mathcal{R}_s$, with $3\gamma<|s|<\frac{\pi}{2}-3\gamma$, we have

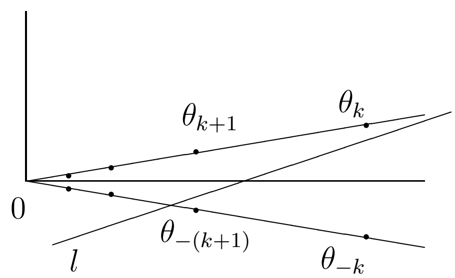
$$\left|\sum_{0<|k|\le m,\,\theta_k\in R}\mathrm{sign}\,(k)\right|\le 2,\quad m=1,2,\cdots.\tag{2.6}$$

Proof. First we note that if l is a line on the plane, then

$$l \cap [\theta_k, \theta_{-k}] \neq \varnothing,\ l \cap [\theta_{k+1}, \theta_{-(k+1)}] \neq \varnothing \tag{2.7}$$

implies

$$\arg l < 3\gamma.$$



Indeed, using a simple geometry, one can check that $\arg(\theta_{-k}\theta_{k+1}) < 3\gamma$. Hence we get $\arg l \leq \arg(\theta_{-k}\theta_{k+1}) < 3\gamma$. Now consider a rectangle

$$R \in \mathcal{R}_s, \quad 3\gamma < |s| < \frac{\pi}{2} - 3\gamma. \tag{2.8}$$

Let us show that

$$\text{if } \theta_n, \theta_{n+1}, \theta_{n+2} \in R, \text{ then } \theta_{-(n+1)} \in R. \tag{2.9}$$

Suppose we have the converse $\theta_{-(n+1)} \notin R$. Then we can determine a line l containing a side of R and separating the points $\theta_n, \theta_{n+1}, \theta_{n+2}$ from $\theta_{-(n+1)}$. Obviously we shall have

$$l \cap [\theta_{n+1}, \theta_{-(n+1)}] \neq \varnothing,$$

and one of two following relations: $l \cap [\theta_n, \theta_{-n}] \neq \varnothing$ or $l \cap [\theta_{n+2}, \theta_{-(n+2)}] \neq \varnothing$. So we have (2.7) for $k = n$ or $n+1$ and therefore $\arg l < 3\gamma$, which is a contradiction with (2.8). Similarly

$$\text{if } \theta_{-n}, \theta_{-(n+1)}, \theta_{-(n+2)} \in R, \text{ then } \theta_{n+1} \in R. \tag{2.10}$$

Now let p and q are the numbers of elements of the sets $\{1 \leq k \leq m : \theta_k \in R\}$ and $\{-m \leq k \leq -1 : \theta_k \in R\}$. From (2.9) and (2.10) we conclude $|p - q| \leq 2$, which implies (2.6).

$\square$

Lemma 3. *For any numbers $0<\varepsilon<1$ and $0<\gamma\le\frac{\pi}{12}$ there exists a bounded function $\phi(x)=\phi(x_1,x_2)$ defined on $\mathbb{R}^2$ such that*

$$\operatorname{supp}\phi\subset B(\varepsilon),\quad \int_{\mathbb{R}^2}\phi(x)dx=0,\quad \int_{\mathbb{R}^2}|\phi(x)|dx\le 1, \tag{2.11}$$

$$\int_{\operatorname{rot}_s\big([0,x_1]\times[0,x_2]\big)}\phi(x)dx\ge\frac{1}{4},\ \text{if}\quad x_1,x_2\ge\varepsilon,\quad |s|\le\gamma, \tag{2.12}$$

$$M_s\phi(x)<\varepsilon,\ \text{as } x\notin\Gamma_s(2\varepsilon)\cup\Gamma_{s^\perp}(2\varepsilon),\ 3\gamma<|s|<\frac{\pi}{2}-3\gamma. \tag{2.13}$$

Proof. Consider the sequence $\theta=\theta^+\cup\theta^-$ where

$$\begin{aligned}\theta^+&=\{\theta_k:\quad k=1,2,\cdots,N\},\\ \theta^-&=\{\theta_k:\quad k=-1,-2,\cdots,-N\},\ N=[10\varepsilon^{-3}]+1,\end{aligned} \tag{2.14}$$

and θ_k are defined in (2.5). We have

$$\theta_k\in B\left(\frac{\varepsilon}{\sqrt{2}}\right)\subset B(\varepsilon),\quad \theta_k\in\{x:x_2=\operatorname{tg}\gamma\cdot x_1\}\quad k=\pm1,\pm2,\cdots.$$

Define the balls b_k, denoting

$$b_k=\{x\in R^2:|x-\theta_k|<r\},\quad k=\pm1,\pm2,\cdots,\pm N.$$

Choosing a small number $r>0$, we provide the following conditions:

1) $b_k\subset B(\varepsilon)$ and they are mutually disjoint,
2) if $k>0$, then b_k is in the upper half-plane, if $k<0$ is in lower,
3) any line l with $|\arg l|\ge 3\gamma$ intersects at most two b_k.

We define

$$\phi(x)=\frac{1}{2\pi Nr^2}\sum_{k=1}^N\big(\mathbb{I}_{b_k}(x)+\mathbb{I}_{b_{-k}}(x)\big),$$

where $\mathbb{I}_{b_k}$ is the characteristic function of b_k. The conditions (2.11) are clear. To show (2.12) we shall use conditions 1) and 2). We fix numbers $x_1,x_2>\varepsilon$. If $0\le s<\gamma$, then we have

$$\operatorname{rot}_s\big([0,x_1]\times[0,x_1]\big)\cap b_k=\varnothing \text{ as } -N\le k<0,$$
$$|\operatorname{rot}_s\big([0,x_1]\times[0,x_2]\big)\cap b_k|>\frac{|b_k|}{2}=\frac{\pi r^2}{2}\text{ as } 0<k\le N.$$

Therefore

$$\int_{\operatorname{rot}_s\big([0,x_1]\times[0,x_2]\big)}\phi(x)dx=\frac{1}{2\pi Nr^2}\sum_{k=1}^N\int_{\operatorname{rot}_s\big([0,x_1]\times[0,x_2]\big)}\mathbb{I}_{b_k}(x)dx\ge\frac{1}{4}.$$

If $-\gamma<s\le 0$, then

$$b_k\subset\operatorname{rot}_s\big([0,x_1]\times[0,x_1]\big),\quad k>0,$$
$$|\operatorname{rot}_s\big([0,x_1]\times[0,x_2]\big)\cap b_k|\le\frac{|b_k|}{2}=\frac{\pi r^2}{2},\quad k>0,$$

and then similarly we obtain (2.12). We shall prove now if

$$R \in \mathcal{R}_s,\ 3\gamma < |s| < \frac{\pi}{2} - 3\gamma \tag{2.15}$$

then

$$\left|\int_R \phi(x)dx\right| \le \frac{10}{N} < \varepsilon^3. \tag{2.16}$$

We have

$$\int_R \phi(x)dx = \frac{1}{2\pi N r^2} \sum_{b_k \cap R \neq \varnothing} \int_R \mathbb{I}_{b_k}(x)dx = \tag{2.17}$$

$$\frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_R \mathbb{I}_{b_k}(x)dx + \frac{1}{2\pi N r^2} \sum_{\theta_k \notin R, b_k \cap R \neq \varnothing} \int_R \mathbb{I}_{b_k}(x)dx.$$

The conditions $\theta_k \notin R, b_k \cap R \neq \varnothing$ mean that b_k intersects a side of R. Also we have that if a line l contains a side of R then $|\arg l| > 3\gamma$. On the other hand by the condition 3) any line with $|\arg l| > 3\gamma$ can intersect not more than two balls b_k. So the number of terms in the second sum doesn't exceed 8. Therefore

$$\left|\frac{1}{2\pi N r^2} \sum_{\theta_k \notin R, b_k \cap R \neq \varnothing} \int_R \mathbb{I}_{b_k}(x)dx\right| \le \frac{4}{N}. \tag{2.18}$$

By the same reason the equality

$$\int_R \mathbb{I}_{b_k}(x)dx = \int_{\mathbb{R}^2} \mathbb{I}_{b_k}(x)dx$$

fails for not more than 8 different k's. Therefore

$$\left|\frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_R \mathbb{I}_{b_k}(x)dx - \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_{\mathbb{R}^2} \mathbb{I}_{b_k}(x)dx\right| \le \frac{4}{N}.$$

Hence we obtain

$$\left|\frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_R \mathbb{I}_{b_k}(x)dx\right| \le \left|\frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_{\mathbb{R}^2} \mathbb{I}_{b_k}(x)dx\right| + \frac{4}{N} = \tag{2.19}$$

$$\left|\frac{1}{2N} \sum_{\theta_k \in R} \operatorname{sign}(k)\right| + \frac{4}{N} \le \frac{5}{N},$$

where the last inequality follows from the Lemma 2. Combining (2.17), (2.19) and (2.18) we get (2.16). Fix a slope s with $3\gamma < |s| \le \frac{\pi}{4}$ and take a point $x \in \mathbb{R}^2$ such that

$$x \notin \Gamma_s(2\varepsilon) \cup \Gamma_{s^\perp}(2\varepsilon),$$
$$x \in R \in \mathcal{R}_s,\ 3\gamma < |s| < \frac{\pi}{2} - 3\gamma.$$

We need to prove

$$\frac{1}{|R|}\int_R \phi(t)dt \le \varepsilon. \tag{2.20}$$

Assume the lengths of the sides of R are a and b. If R doesn't contain a point θ_k then (2.20) is trivial. So we suppose there exists at least one point $\theta_k \in R$. Hence R has an intersection with $B(\varepsilon)$ and $\big(\Gamma_s(2\varepsilon)\cup\Gamma_{s^\perp}(2\varepsilon)\big)^c$. Taking account of $R \in \mathcal{R}_s$ we get $a, b > \varepsilon$. Hence by (2.16) we get

$$\frac{1}{|R|}\int_R \phi(t)dt \le \frac{\varepsilon^3}{ab} \le \varepsilon$$

□

Lemma 4. *For any numbers $0<\varepsilon,\delta<1/10$, and interval $S=[\alpha-\gamma,\alpha+\gamma]\subset[0,\pi/2)$ with $0<\gamma\le\frac{\pi}{12}$ there exist a bounded function $\phi(x)$ and numbers ν,ν' with $0<\nu<\nu'$ such that*

$$\sup_{k\in\mathbb{Z}^2}\int_{Q_k}|\phi(x)|dx \le 1 \tag{2.21}$$

$$\text{mes}^*\{x\in\mathbb{R}^2:\ M_s\phi(x)>\varepsilon\}<\varepsilon, \quad 3\gamma<|s-\alpha|<\frac{\pi}{2}-3\gamma, \tag{2.22}$$

$$\text{mes}^*\{x\in\mathbb{R}^2:\ M_s^{[0,\nu)}\phi(x)>\varepsilon\}<\varepsilon, \quad s\in[0,2\pi), \tag{2.23}$$

$$M_s^{[\nu',\infty)}\phi(x)<\varepsilon, \quad x\in\mathbb{R}^2,\ s\in[0,2\pi), \tag{2.24}$$

$$\text{mes}_*\{M_s^{[\nu,\nu']}\phi(x)>\frac{1}{\delta}\}>\frac{\delta}{4}\ln\frac{1}{12\delta}, \quad s\in S. \tag{2.25}$$

Proof. Without loss of generality we may assume $\alpha=0$, i.e. $S=[-\gamma,\gamma]$. We take $\lambda=\min\{\varepsilon/100,\delta\}$ and consider a double sequence $\varepsilon_k=\varepsilon_{k_1,k_2}=\lambda 2^{-(|k_1|+|k_2|)}$, $k\in\mathbb{Z}^2$. Using Lemma 3 we can find functions $\phi_k(x)$ with following conditions:

$$\operatorname{supp}\phi_k\subset B(\varepsilon_k)\subset B(\varepsilon), \tag{2.26}$$

$$\int_{Q_0}\phi_k(x)dx=0, \quad \int_{Q_0}|\phi_k(x)|dx\le 1, \tag{2.27}$$

$$\int_{\text{rot}_s(R_x)}\phi_k(x)dx>\frac{1}{4},\ R_x=[0,x_1]\times[0,x_2],\ x_1,x_2\ge\delta\ge\varepsilon_k,\ |s|<\gamma, \tag{2.28}$$

$$M_s\phi_k(x)<\varepsilon_k,\ \text{as } x\notin\Gamma_s(2\varepsilon_k)\cup\Gamma_{s^\perp}(2\varepsilon_k),\ 3\gamma<|s|\le\frac{\pi}{2}-3\gamma, \tag{2.29}$$

where $k=(k_1,k_2)$. Denote

$$\phi(x)=\sum_{k\in\mathbb{Z}^2}\phi_k(x+k), \tag{2.30}$$

$$E_s=\bigcup_{k\in\mathbb{Z}^2}\Big(k+\big(\Gamma_s(2\varepsilon_k)\cup\Gamma_{s^\perp}(2\varepsilon_k)\big)\Big). \tag{2.31}$$

We obviously have (2.21) and

$$\operatorname{supp}\phi(x)\subset\bigcup_{k\in\mathbb{Z}^2}\big(k+B(\varepsilon)\big), \tag{2.32}$$

$$\int_{Q_k}\phi(x)dx=0,\quad k\in\mathbb{Z}^2. \tag{2.33}$$

Proof of (2.22): For any square Q_j, $j\in\mathbb{Z}^2$, we have

$$|Q_j\cap\big(k+\Gamma_s(2\varepsilon_k)\big)|\le\operatorname{diam}Q_j\times\operatorname{mes}_s\big(k+\Gamma_s(2\varepsilon_k)\big)=4\varepsilon_k\sqrt{2},$$
$$|Q_j\cap\big(k+\Gamma_{s^\perp}(2\varepsilon_k)\big)|\le 4\varepsilon_k\sqrt{2}.$$

Hence we obtain

$$\operatorname{mes}^*(E_s)\le\sum_k 8\sqrt{2}\varepsilon_k=32\sqrt{2}\lambda\le\varepsilon. \tag{2.34}$$

From (2.29) it follows that

$$M_s\phi_k(x+k)\le\varepsilon_k,\quad x\notin E_s\supset k+\big(\Gamma_s(2\varepsilon_k)\cup\Gamma_{s^\perp}(2\varepsilon_k)\big),\quad 3\gamma<|s|\le\frac{\pi}{2}-3\gamma.$$

Then according (2.30) and (2.31) we get

$$M_s\phi(x)\le\sum_k M_s\phi_k(x+k)\le\sum_k\varepsilon_k\le\varepsilon, x\notin E_s,\quad 3\gamma<|s|\le\frac{\pi}{2}-3\gamma,$$

and combining this with (2.34) we obtain (2.22).

Proof of (2.23): From (2.32) it follows that

$$\lim_{\nu\to 0}M_s^{[0,\nu)}\phi(x)=0,\ \text{ if } x\notin\bigcup_{k\in\mathbb{Z}^2}\big(k+B(\varepsilon)\big),\quad s\in[0,2\pi),$$

therefore for a small $\nu<\delta$ we shall have (2.23), since

$$\operatorname{mes}^*\left(\bigcup_{k\in\mathbb{Z}^2}\big(k+B(\varepsilon)\big)\right)=|B(\varepsilon)|=\pi\varepsilon^2\le\varepsilon.$$

Proof of (2.24): From (2.33) we obtain

$$\lim_{\nu'\to\infty}\int_R\phi(\nu' x)dx=0$$

for any rectangle R and the convergence is uniformly by $R\in\mathcal{R}_s^{[1,\infty)}$, $s\in[0,2\pi)$. So for a big $\nu'>1/4$ we shall have

$$M_s^{[1,\infty)}\phi(\nu' x)<\varepsilon,\ x\in\mathbb{R}^2,\quad s\in[0,2\pi).$$

By dilation we get

$$M_s^{[\nu',\infty)}\phi(x)=M_s^{[1,\infty)}\phi(\nu' x)<\varepsilon,\ x\in\mathbb{R}^2,\quad s\in[0,2\pi),$$

which gives (2.24).

Proof of (2.25): Consider the set

$$A=\Big\{x=(x_1,x_2):\ x_1x_2\le\frac{\delta}{4},\quad \delta\le x_1,x_2\le\frac{1}{4}\Big\}. \tag{2.35}$$

We have

$$\operatorname{rot}_s A \subset B\left(\frac{1}{2}\right), \quad s \in [-\pi/4, \pi/4), \tag{2.36}$$

$$|A| = \int_\delta^{1/4} \frac{\delta}{4t} dt - \delta\left(\frac{1}{4} - \delta\right) > \frac{\delta}{4} \ln \frac{1}{12\delta} \tag{2.37}$$

If $x = (x_1, x_2) \in A$, then

$$\frac{1}{4} \geq x_1, x_2 \geq \delta > \varepsilon_k, \quad |R_x| \leq \frac{\delta}{4} \tag{2.38}$$

So by (2.28)

$$\int_{k+\operatorname{rot}_s(R_x)} \phi_k(k+t)dt = \int_{\operatorname{rot}_s(R_x)} \phi_k(t)dt > \frac{1}{4}, \text{ as } x \in A,\ |s| < \gamma,$$

and therefore from (2.26) we can get

$$\int_{k+\operatorname{rot}_s(R_x)} \phi(t)dt = \int_{k+\operatorname{rot}_s(R_x)} \phi_k(k+t)dt > \frac{1}{4}, \text{ as } x \in A,\ |s| < \gamma. \tag{2.39}$$

According to $\nu < \delta$, $\nu' > 1/4$ we have $R_x \in \mathcal{R}_0^{[\delta,1/4]} \subset \mathcal{R}_0^{[\nu,\nu']}$. Since $|R_x| \leq \delta/4$ from (2.39) and (2.38) we conclude

$$M_s^{[\nu,\nu']}\phi(x) > \frac{1}{4|R_x|} > \frac{1}{\delta}, \quad x \in G_s = \bigcup_k \left(k + \operatorname{rot}_s A\right), \quad |s| < \gamma. \tag{2.40}$$

In addition, by (2.35), (2.36) and (2.37), for any $m \in \mathbb{Z}^2$ we get

$$|(m + Q_0) \cap G_s| = |m + \operatorname{rot}_s A| = |A| > \frac{\delta}{4} \ln \frac{1}{12\delta},$$

which implies

$$\operatorname{mes}^* G_s > \frac{\delta}{4} \ln \frac{1}{12\delta}.$$

Combining this with (2.40) we obtain (2.25). □

3. Proofs of Theorems

Proof of Theorem 1. Let G be an arbitrary G_δ-set in $[0, \pi/2)$. So

$$G = \left(\cap_{k=1}^\infty G_k\right) \cap [0, \pi/2),$$

where $G_k \subset \mathbb{R}$ are open sets and

$$G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots .$$

Each G_k is union of a mutually disjoint intervals, i.e.

$$G_k = \cup_j I_j^k.$$

We note that an arbitrary interval $I = (\alpha, \beta) \subset \mathbb{R}$ can be split to disjoint intervals $I_i = [\alpha_i, \beta_i)$ such that

$$|I_i| \leq \frac{\pi}{12}, \quad 3I_i \subset I, \quad \sum_i \mathbb{I}_{3I_i}(x) \leq 8.$$

For $I=(-1,1)$ such a partition is

$$\left[1-\left(\frac{9}{10}\right)^k, 1-\left(\frac{9}{10}\right)^{k+1}\right), \quad k=0,1,2,\cdots,$$
$$\left[\left(\frac{9}{10}\right)^{k+1}-1, \left(\frac{9}{10}\right)^{k}-1\right), \quad k=0,1,2,\cdots,$$

We do a similar splitting for any I_j^k. Let J_t, $t=1,2,\cdots$, be a numeration of those splitting intervals J for wich $J\cap[0,\pi/2)\neq\varnothing$. We denote $l_t=J_t\cap[0,\pi/2)$. It is easy to check the following two relations

1) if $x\in G$, then x belongs to infinite number of l_t's,

2) if $x\notin G$ then x belongs only to finite number of $3l_t$'s.

We chose integers $0=m_0<m_1<m_2<\cdots$ satisfying

$$\prod_{k=m_t+1}^{m_{t+1}}\left(1-\frac{1}{k\ln k}\right)<\frac{1}{2^t}, \quad t=1,2,\cdots. \tag{3.1}$$

We denote

$$S_k=l_t, \text{ if } m_t<k\leq m_{t+1}. \tag{3.2}$$

Using Lemma 4 for $S=S_k$, $\varepsilon=1/2^k$, $\delta=1/k\ln^2 k$, we may define functions $\phi_k(x)$ and numbers $0<\nu_k<\nu_k'$ with conditions (2.21)-(2.25). We denote

$$U_{s,k}=\{x\in\mathbb{R}^2:\ M_s\phi_k(x)\leq\frac{1}{2^k}\}, \tag{3.3}$$

$$V_{s,k}'=\{x\in\mathbb{R}^2:\ M_s^{[0,\nu_k)}\phi_k(x)\leq\frac{1}{2^k}\}, \tag{3.4}$$

$$V_{s,k}''=\{x\in\mathbb{R}^2: M_s^{[\nu_k,\nu_k']}\phi_k(x)>k\ln^2 k\}. \tag{3.5}$$

By (2.22),(2.23),(2.25) we have

$$\text{mes}_* U_{s,k}>1-\frac{1}{2^k}, \quad s\in[0,\pi/2)\setminus 3S_k, \tag{3.6}$$

(we may replace the condition $3\gamma<|s-\alpha|<\frac{\pi}{2}-3\gamma$ in (2.22) by $s\in[0,\pi/2)\setminus 3S$ because the second implies the first) and

$$\text{mes}_* V_{s,k}'>1-\frac{1}{2^k}, \quad s\in[0,\pi/2), \tag{3.7}$$

$$\text{mes}_* V_{s,k}''>\frac{1}{4k\ln^2 k}\ln\frac{k\ln^2 k}{12}>\frac{c}{k\ln k}, \quad s\in S_k \quad (k\geq 3). \tag{3.8}$$

From (2.24) we get

$$M_s^{[\nu_k',\infty)}\phi_k(x)<\frac{1}{2^k}, \quad x\in\mathbb{R}^2,\ s\in[0,\pi/2). \tag{3.9}$$

We define integers $1=n_0<n_1<n_2<\cdots$, so that

$$\frac{n_k}{n_{k-1}}>\max\left(\frac{4}{\nu_{k-1}},\frac{\nu_k'}{\nu_{k-1}}\right), \quad k=1,2, \quad , \tag{3.10}$$

and denote $\mu_k=\nu_k/n_k$. It is clear

$$\mu_{k-1}>\frac{\nu_k'}{n_k}>\mu_k,\quad k=2,3,\cdots .$$

Consider the functions

$$\psi_k(x)=\phi_k(n_kx),\quad x\in Q_0. \tag{3.11}$$

According to (3.3)-(3.5) and (3.11), we obviously have

$$M_s\psi_k(x)\le\frac{1}{2^k},\quad x\in \operatorname{dil}_{n_k}U_{s,k},\quad s\in[0,\pi/2)\setminus 3S_k, \tag{3.12}$$

$$M_s^{[0,\mu_k)}\psi_k(x)=M_s^{[0,\nu_k/n_k)}\psi_k(x)\le\frac{1}{2^k}\quad x\in \operatorname{dil}_{n_k}V'_{s,k},\quad s\in[0,\pi/2), \tag{3.13}$$

$$M_s^{[\mu_k,\mu_{k-1}]}\psi_k(x)>M_s^{[\nu_k/n_k,\nu_k'/n_k]}\psi_k(x)>k\ln^2k,\quad x\in \operatorname{dil}_{n_k}V''_{s,k},\quad s\in S_k, \tag{3.14}$$

$$M_s^{[\mu_{k-1},\infty)}\psi_k(x)\le M_s^{[\nu_k'/n_k,\infty)}\psi_k(x)\le\frac{1}{2^k},\quad x\in\mathbb{R}^2,\quad s\in[0,\pi/2). \tag{3.15}$$

Desired function will be

$$f(x)=\sum_{k=1}^\infty\frac{\psi_k(x)}{k\ln^{3/2}k},\quad x\in Q_0. \tag{3.16}$$

Denote

$$U_s=\limsup_{k\to\infty}\Big(\big(\operatorname{dil}_{n_k}U_{s,k}\big)\cap Q_0\Big), \tag{3.17}$$

where $\limsup_{k\to\infty}A_k$ means $\cup_n\cap_{k\ge n}A_k$. If $s\notin G$, then by 2) $s\in[0,\pi/2)\setminus 3S_k$ as $k>k(s)$. Therefore, by (3.6) we have $|\operatorname{dil}_{n_k}U_{s,k}\cap Q_0|\ge \operatorname{mes}_*U_{s,k}>1-1/2^k$, $k>k(s)$, and so we get

$$|U_s|=1 \text{ if } s\notin G. \tag{3.18}$$

From (3.12) and (3.17) we get, that for any $x\in U_s$

$$M_s\psi_k(x)\le\frac{1}{2^k},\quad k>k(x).$$

Hence, if $\varepsilon>0$, then for an appropriate $N>k(x)$ we get

$$M_s\bigg(\sum_{k=N+1}^\infty\frac{\psi_k(x)}{k\ln^{3/2}k}\bigg)\le\sum_{k=N+1}^\infty\frac{M_s\psi_k(x)}{k\ln^{3/2}k}\le\sum_{k=N+1}^\infty\frac{1}{k2^k\ln^{3/2}k}<\varepsilon. \tag{3.19}$$

On the other hand, since

$$\sum_{k=1}^N\frac{\psi_k(x)}{k\ln^{3/2}k}$$

is a bounded function, the basis $\mathcal{R}_s$ differentiates its integral. So, taking account of (3.19) and (3.16) we get $\int f$ differentiable by $\mathcal{R}_s$ if $s\in[0,\pi/2)\setminus G$.

Now let us take $s\in G$. We have $s\in l_{t_i}$, $i=1,2,\cdots$. Hence $s\in S_k$ if $m_{t_i}<k\le m_{t_i+1}$, $i=1,2,\cdots$. We notice, that each $V''_{s,k}$ defined in (3.5) is

ν_k-set, and by (3.10) $n_{k+1} > \frac{4}{\nu_k} n_k$. Therefore, using (3.1), from Lemma 1 we obtain

$$\left| \bigcup_{k=m_{t_i}+1}^{m_{t_i+1}} \operatorname{dil}_{n_k} V''_{s,k} \cap Q_0 \right| \geq 1 - \prod_{k=m_{t_i}+1}^{m_{t_i+1}} \left(1 - \frac{1}{k \ln k}\right) > 1 - \frac{1}{2^t}. \tag{3.20}$$

Denoting

$$V_s = \left(\limsup_{k\to\infty} \operatorname{dil}_{n_k} V'_{s,k}\right) \bigcap \left(\limsup_{i\to\infty} \bigcup_{k=m_{t_i}+1}^{m_{t_i+1}} \operatorname{dil}_{n_k} V''_{s,k}\right) \bigcap Q_0,$$

from (3.20) and (3.7) we get

$$|V_s| = 1, \quad s \in G. \tag{3.21}$$

On the other hand if $x \in V_s$, then

$$x \in \operatorname{dil}_{n_{k_i}} V''_{s,k_i}, \quad i = 1, 2, \cdots,$$
$$x \in \operatorname{dil}_{n_k} V'_{s,k}, \quad k > k(x).$$

where $k_i \to \infty$, and therefore, by (3.13) and (3.15) we have

$$M_s^{[\mu_{k_i}, \mu_{k_i-1}]} \psi_j(x) \leq \frac{1}{2^{k_i}}, \text{ if } j \neq k_i.$$

The case $j > k_i$ follows from (3.15) and $j < k_i$ from (3.13). From (3.14) we get

$$M_s^{[\mu_{k_i}, \mu_{k_i-1}]} \psi_{k_i}(x) > k_i \ln^2 k_i.$$

So if $k_i > k(x)$, then

$$M_s f(x) \geq M_s^{[\mu_{k_i}, \mu_{k_i-1}]} f(x) \geq$$
$$\frac{M_s^{[\mu_{k_i}, \mu_{k_i-1}]} \psi_{k_i}(x)}{k_i \ln^{3/2} k_i} - \sum_{j \neq k_i} \frac{M_s^{[\mu_{k_i}, \mu_{k_i-1}]} \psi_j(x)}{j \ln^{3/2} j} \geq c\sqrt{\ln k_i} - \sum_{j \neq k_i} \frac{1}{j 2^j \ln^{3/2} j}$$

and so $\overline{D}_s f(x) = \infty$, whenever $x \in V_s$ and $s \in G$. Since $|V_s| = 1$ by (3.21), the theorem is completely proved. □

Proof of Theorem 2. The necessity of the theorem is shown in the introduction. To prove the sufficiency we let $V \in [0, \pi/2)$ to be an arbitrary $G_{\delta\sigma}$ set and we have

$$V = \bigcup_n V_n$$

where each V_n is G_δ. According to Theorem 1 for each V_n there exists a function $f_n \in L^1(\mathbb{R}^2)$ such that its integral differentiable by $\mathcal{R}_s$ as $s \notin V_n$ and $\overline{D}_s f_n(x) = \infty$ a.e. if $s \in V_n$. Denote $g_n(x) = \chi_{Q_n}(x) f_n(x)$, where Q_n is

a family of arbitrary pairwise disjoint unit open squares, and consider the function

$$g(x)=\sum_{n=1}^{\infty}g_n(x).$$

Since the supports of the functions g_n are disjoint for any point $x\in Q_n$ and any s we have

$$\overline{D}_s g(x)=\overline{D}_s g_n(x)=\overline{D}_s f_n(x).$$

If $s\in V$ then $s\in V_n$ for some n. So we get $\overline{D}_s g(x)=\overline{D}_s f_n(x)=\infty$ almost everywhere on the square Q_n. Using disjointness of the supports of the functions g_n once again we conclude that if $s\notin V$ then

$$\lim_{d(R)\to 0, x\in R\in\mathcal{R}_s}\frac{1}{|R|}\int_R g=g(x)\text{ a.e. }.$$

Finally we get that V is WR-set and Theorem 2 is proved. $\square$

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